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Applied Mathematics Letters 13 (2000) 101–106

Applied Mathematics Letters

www.elsevier.nl/locate/aml

# Some Combinatorial Series Identities Associated with the Digamma Function and Harmonic Numbers

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*(Received and accepted April 1999)*

**Abstract**—The authors developed closed-form sums of several interesting families of series associated with the Digamma (or Psi) function and harmonic numbers. A number of illustrative examples and applications of the main results are also considered. 

**Keywords**—Digamma (or Psi) function, Harmonic numbers, Gamma function, Combinatorial (or binomial) coefficients, Series identities, Beta function, Gauss summation theorem, Chu-Vandermonde summation theorem, Gauss hypergeometric series.

## 1. INTRODUCTION AND PRELIMINARIES

In terms of the familiar Gamma function, the Digamma (or Psi) function $\psi(z)$ and the Beta function $B(\alpha, \beta)$ are defined by (cf., [1, Chapter 1])

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1}$$

and

$$B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2}$$

The present investigation was initiated during H. M. Srivastava's visits to the Institute of Mathematics (Academia Sinica) at Taipei, National Chiao Tung University at Hsin-Chu, Soochow University at Taipei, and Chung Yuan Christian University at Chung-Li in February 1998.
This work was supported, in part, by the National Science Council of the Republic of China under Grant NSC 87-2119-M-146-001 and the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.


PII: S0893-9659(99)00193-7

respectively. Also let the combinatorial (or binomial) coefficient be defined, in general, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)\Gamma(\mu+1)} = \binom{\lambda}{\lambda-\mu}, \qquad (\lambda,\mu\in\mathbb{C}). \tag{3}$$

Then it is easily seen that [1, p. 16, equation 1.7.1 (10)]

$$\begin{gathered} H_n(z) := \sum_{j=1}^{n} \frac{1}{z+j} = \psi(z+n+1) - \psi(z+1), \\ (n\in\mathbb{N} := \{1,2,3,\dots\}; z\in\mathbb{C}\setminus\{-1,-2,-3,\dots\}), \end{gathered} \tag{4}$$

so that the harmonic numbers $H_n$ are given by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} = H_n(0), \qquad (n\in\mathbb{N}). \tag{5}$$

The main object of this paper is to present closed-form expressions for three families of combinatorial series associated with the functions $\psi(z)$ and $H_n(z)$. We also consider several illustrative examples and applications of our main results.

The following known combinatorial series identities will be required in our present investigation (cf. [2]):

$$\sum_{k=0}^{\infty}(-1)^k\binom{\lambda}{k} = 0, \qquad (\mathrm{Re}(\lambda)>0), \tag{6}$$

$$\begin{gathered} \sum_{k=m}^{n}(-1)^k\binom{\lambda}{k} = (-1)^m\binom{\lambda-1}{m-1} + (-1)^n\binom{\lambda-1}{n}, \\ (\lambda\in\mathbb{C}; m,n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}), \end{gathered} \tag{7}$$

which in the special case when $m=0$ and $\lambda=n$, would obviously provide a finite-series form of (6);

$$\sum_{k=0}^{\infty}(-1)^k\binom{\lambda}{k}\frac{1}{\mu+k} = B(\lambda+1,\mu) = \frac{1}{\mu}\binom{\lambda+\mu}{\lambda}^{-1}, \tag{8}$$

$$(\mathrm{Re}(\lambda)>-1; \mu\in\mathbb{C}\setminus\{0,-1,-2,\dots\}),$$

$$\begin{gathered} \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\mu+k}{\nu}^{-1} = \frac{\nu}{\nu+n}\binom{\mu+n}{\mu-\nu}^{-1}, \\ (\nu\in\mathbb{C}\setminus\{0,-1,-2,\dots; \mu-\nu\neq-1,-2,-3,\dots\}). \end{gathered} \tag{9}$$

The last combinatorial series identities (8) and (9) are contained, respectively, in the Gauss summation theorem [1, p. 104, equation 2.8 (46)]

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad (\mathrm{Re}\,(c-a-b)>0; c\neq 0,-1,-2,\dots) \tag{10}$$

and its special case when $a=-n$ $(n\in\mathbb{N}_0)$, known as the Chu-Vandermonde summation theorem

$${}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}, \qquad (n\in\mathbb{N}_0; c\neq 0,-1,-2,\dots), \tag{11}$$

where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric series defined by

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{12}$$

in terms of the Pochhammer symbol $(\lambda)_k$ given by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & (k=0), \\ \lambda(\lambda+1)\dots(\lambda+k-1), & (k \in \mathbb{N}). \end{cases} \tag{13}$$

As a matter of fact, by applying (10) instead of its special case (11), we can readily obtain a unification and generalization of both (8) and (9) in the form

$$\begin{gathered} \sum_{k=0}^{\infty}(-1)^k \binom{\lambda}{k} \binom{\mu+k}{\nu}^{-1} = \frac{\nu}{\nu+\lambda} \binom{\lambda+\mu}{\mu-\nu}^{-1}, \\ (\operatorname{Re}(\nu+\lambda) > 0; \mu \in \mathbb{C} \setminus \{0,-1,-2,\dots\}), \end{gathered} \tag{14}$$

which yields (8) when $\nu = 1$ and (9) when $\lambda = n$ $(n \in \mathbb{N}_0)$. For several general combinatorial series identities stemming essentially from (10) and (11), one may refer to a recent paper by Gould and Srivastava [3].

## 2. THE MAIN RESULTS

We begin by stating one of our main results.

THEOREM 1. *Let the function $H_n(z)$ be defined by (4). Then*

$$\begin{gathered} \sum_{k=0}^{\infty}(-1)^k \binom{\lambda}{k} H_{n+k}(\mu) = -B(\lambda, \mu+n+1), \\ (\operatorname{Re}(\lambda) > 0; n \in \mathbb{N}; \mu \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}). \end{gathered} \tag{15}$$

PROOF. Denote, for convenience, the first member of the combinatorial series identity (15) by $\Theta(\lambda, \mu)$. Then, by appealing appropriately to definition (4) and the known result (6), we find that

$$\begin{aligned} \Theta(\lambda,\mu) :&= \sum_{k=0}^{\infty}(-1)^k \binom{\lambda}{k} H_{n+k}(\mu) \\ &= \sum_{k=0}^{\infty}(-1)^k \binom{\lambda}{k} [H_n(\mu) + H_k(\mu+n)] \\ &= \sum_{k=1}^{\infty}(-1)^k \binom{\lambda}{k} \sum_{j=1}^{k} \frac{1}{\mu+n+j} \\ &= \sum_{j=1}^{\infty} \frac{1}{\mu+n+j} \sum_{k=j}^{\infty}(-1)^k \binom{\lambda}{k} \\ &= -\sum_{j=1}^{\infty} \frac{1}{\mu+n+j} \sum_{k=0}^{j-1}(-1)^k \binom{\lambda}{k} \\ &= -\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\mu+n+j} \binom{\lambda-1}{j-1}, \end{aligned} \tag{16}$$

where we have also applied the series identity (7) with $m = 0$ and $n = j - 1$ $(j \in \mathbb{N})$.

Finally, we make use of the series identity (8), and we find from (16) that

$$\Theta(\lambda,\mu) = -\sum_{j=0}^{\infty}(-1)^j \binom{\lambda-1}{j} \frac{1}{\mu+n+j+1} = -B(\lambda,\mu+n+1), \tag{17}$$

which evidently proves assertion (15) under the parametric constraints stated already.

Next, we prove the following theorem.

THEOREM 2. *Let $H_n$ denote the harmonic numbers defined by (5). Then*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{H_{n+k}}{n+k}=(H_{m+n}-H_m)\,B(m+1,n),\qquad (m\in\mathbb{N}_0;n\in\mathbb{N}),\tag{18}$$

*or equivalently,*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{H_{n+k}}{n+k}=[\psi(m+n+1)-\psi(m+1)]\,B(m+1,n),\qquad (m\in\mathbb{N}_0;n\in\mathbb{N}).\tag{19}$$

PROOF. In view of the special case $\lambda=m$ $(m\in\mathbb{N}_0)$ of the well-known combinatorial identity

$$\binom{\lambda+1}{k}=\binom{\lambda}{k}+\binom{\lambda}{k-1},\qquad (\lambda\in\mathbb{C};k\in\mathbb{N}_0),\tag{20}$$

assertion (18) can easily be proven by appealing to the principle of mathematical induction on $m$. And the equivalent form (19) would follow readily from (18) by means of the relationship (4) with, of course, $z=0$.

Finally, we have the following.

THEOREM 3. *Let $B(\alpha,\beta)$ denote the Beta function defined by (2). Then*

$$\sum_{k=1}^{\infty}(-1)^k\binom{\lambda}{k}[B(\mu+\ell,k)-B(\mu,k)]=\sum_{j=0}^{\ell-1}\left(\frac{1}{\mu+j}-\frac{1}{\lambda+\mu+j}\right),\tag{21}$$
$$(\mathrm{Re}(\lambda+\mu)>0;\mu\in\mathbb{C}\setminus\{0,-1,-2,\dots\};\ell\in\mathbb{N}),$$

*and (more generally),*

$$\begin{aligned}&\sum_{k=1}^{\infty}(-1)^k\binom{\lambda}{k}[B(\mu+\rho,\nu+k)-B(\mu,\nu+k)]\\&=B(\rho+\mu,\nu)\left[\frac{(\rho+\mu)_\lambda}{(\rho+\mu+\nu)_\lambda}-1\right]-B(\mu,\nu)\left[\frac{(\mu)_\lambda}{(\mu+\nu)_\lambda}-1\right],\\&(\mathrm{Re}(\lambda+\mu)>\max\{0,-\mathrm{Re}(\rho)\};\mu,\nu,\rho\in\mathbb{C}\setminus\{0,-1,-2,\dots\}).\end{aligned}\tag{22}$$

PROOF. First of all, the general result (22) would follow fairly easily if we apply the Gauss summation theorem (10) to each term on the left-hand side of (22).

If we proceed to the limit as $\nu\to0$, we find from the general result (22) that

$$\begin{aligned}&\sum_{k=1}^{\infty}(-1)^k\binom{\lambda}{k}[B(\mu+\rho,k)-B(\mu,k)]\\&=\lim_{\nu\to0}\left\{B(\rho+\mu,\nu)\left[\frac{(\rho+\mu)_\lambda}{(\rho+\mu+\nu)_\lambda}-1\right]-B(\mu,\nu)\left[\frac{(\mu)_\lambda}{(\mu+\nu)_\lambda}-1\right]\right\}\\&=\Gamma(\rho+\mu)\lim_{\nu\to0}\left\{\frac{(\rho+\mu)_\lambda-(\rho+\mu+\nu)_\lambda}{\Gamma(\lambda+\mu+\nu+\rho)/\Gamma(\nu)}\right\}-\Gamma(\mu)\lim_{\nu\to0}\left\{\frac{(\mu)_\lambda-(\mu+\nu)_\lambda}{\Gamma(\mu+\nu+\lambda)/\Gamma(\nu)}\right\}.\end{aligned}\tag{23}$$

Since (cf., [1, Chapter 1])

$$\Gamma(z)\Gamma(1-z)=\pi\csc(\pi z)\quad\text{and}\quad\psi(1-z)-\psi(z)=\pi\cot(\pi z),\qquad(z\in\mathbb{C}\setminus\mathbb{Z}),\tag{24}$$

it is readily observed that

$$\frac{\psi(1-z)}{\Gamma(1-z)}=\pi^{-1}\Gamma'(z)\sin(\pi z)+\Gamma(z)\cos(\pi z),\qquad(z\in\mathbb{C}\setminus\mathbb{Z}),\tag{25}$$

which immediately yields the limit relationship

$$\lim_{z\to n}\left\{\frac{\psi(1-z)}{\Gamma(1-z)}\right\}=\Gamma(n)\cos(n\pi)=(-1)^n(n-1)!,\qquad (n\in\mathbb{N}). \tag{26}$$

Now we make use of l'Hôpital's rule in (23) and apply the limit relationship (26) with $n=1$ (and $z=1-\nu$). We thus obtain

$$\begin{gathered}\sum_{k=1}^{\infty}(-1)^k\binom{\lambda}{k}[B(\mu+\rho,k)-B(\mu,k)]\\ =[\psi(\mu+\rho)-\psi(\mu)]-[\psi(\lambda+\mu+\rho)-\psi(\lambda+\mu)],\\ (\mathrm{Re}(\lambda+\mu)>\max\{0,-\mathrm{Re}(\rho)\};\mu,\rho\in\mathbb{C}\setminus\{0,-1,-2,\dots\}),\end{gathered}\tag{27}$$

which, in the special case when $\rho=\ell$, $(\ell\in\mathbb{N})$, leads us to assertion (21) by means of (4).

Alternatively, assertion (21) can be proven *directly* (that is, without recourse to the above limit process) by appealing to the following *well-known* (rather *classical*) result in the theory of the Psi (or Digamma) function $\psi(z)$ (see, e.g., [4, p. 126, Entry (6.6.34)] and the references cited there)

$$\sum_{k=1}^{\infty}\frac{(\alpha)_k}{k\,(\gamma)_k}=\psi(\gamma)-\psi(\gamma-\alpha),\qquad (\mathrm{Re}(\gamma-\alpha)>0;\ \gamma\neq 0,-1,-2,\dots). \tag{28}$$

It may be remarked in passing that various special cases and consequences of (28) were revived in many recent works (or serendipities) of fractional calculus, especially in the area of summation of infinite series, as illustrations emphasizing the usefulness of the fractional calculus techniques. For a reasonably detailed *historical* account of the summation formula (28), and also of its numerous consequences and generalizations, one may refer to a recent work on the subject by Nishimoto and Srivastava [5], who furnished many relevant *earlier* references on summation of infinite series by means of fractional calculus.

## 3. ILLUSTRATIVE EXAMPLES AND APPLICATIONS

First of all, since

$$B(\alpha,\beta)=2\int_0^{\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta,\ d\theta,\qquad (\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0), \tag{29}$$

Theorem 1 yields the relationship

$$\begin{gathered}\int_0^{\pi/2}\sin^{2\lambda-1}\theta\,\cos^{2\mu+2n+1}\theta\,d\theta=-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\binom{\lambda}{k}H_{n+k}(\mu),\\ (\min\{\mathrm{Re}(\lambda),\mathrm{Re}(\mu)\}>0;n\in\mathbb{N}).\end{gathered}\tag{30}$$

For $\lambda=n$ $(n\in\mathbb{N})$ and $\mu=0$, assertion (15) reduces immediately to the form

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{n+k}=-\frac{(n-1)!\sqrt{\pi}}{2^{2n}\Gamma(n+1/2)}=-2^{-2n}B\left(n,\left(\frac{1}{2}\right)\right),\qquad (n\in\mathbb{N}). \tag{31}$$

Next, we recall that (cf., [6, p. 538])

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ell\mathrm{n}\,t\,dt=B(\alpha,\beta)[\psi(\beta)-\psi(\alpha+\beta)],\qquad (\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0), \tag{32}$$

which, in conjunction with assertion (19) of Theorem 2, yields the relationship

$$\int_0^1 t^m(1-t)^{n-1}\ell\mathrm{n}\,t\,dt=-\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{H_{n+k}}{n+k},\qquad (m\in\mathbb{N}_0;\ n\in\mathbb{N}). \tag{33}$$

Numerous further illustrations and applications of the assertions of Theorems 1–3 can be given in a similar manner. The details involved are being left as an exercise for the interested reader.

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